Productivity of Interaction Nets – Part 2

Presentation to University of Sussex FOSS Group

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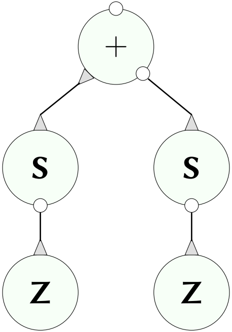
19th November, 2025

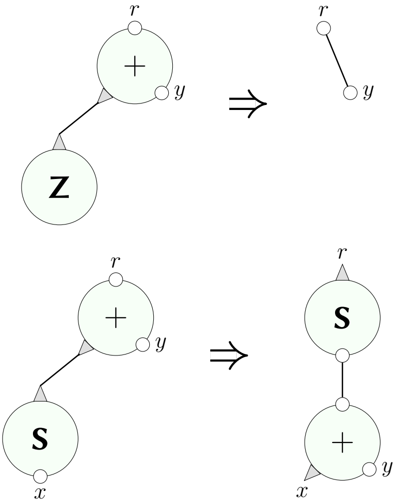
Previously...

Interaction nets



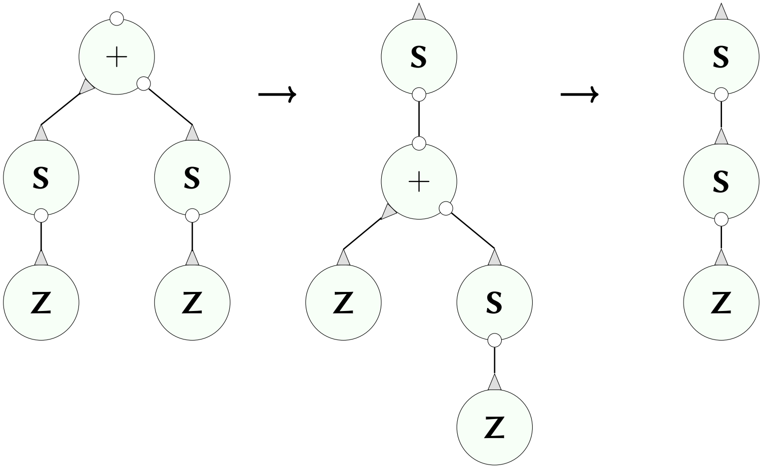
Port graphs = nodes + wires.
Nodes drawn from finite signature: label, arity.
No partitioning of labels.
May be cyclic.
May be disjoint.

Interaction rules



Deterministic depth-1 rules.
Rules preserve interface.

Interaction nets

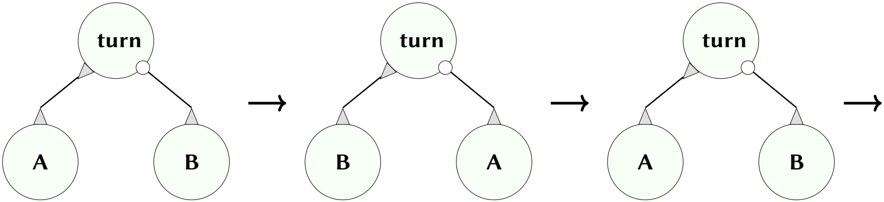


Evaluation $\Downarrow = \xrightarrow{\mathcal{R}}^{*} = \mathsf{lfp}(\xrightarrow{\mathcal{R}})$

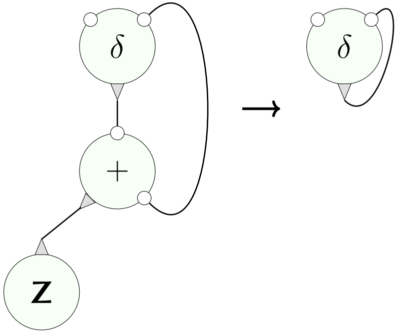
Result is normal form.

$\Downarrow$ is partial endofunction ; domain=co-domain=$\mathcal{N}_\Sigma$.

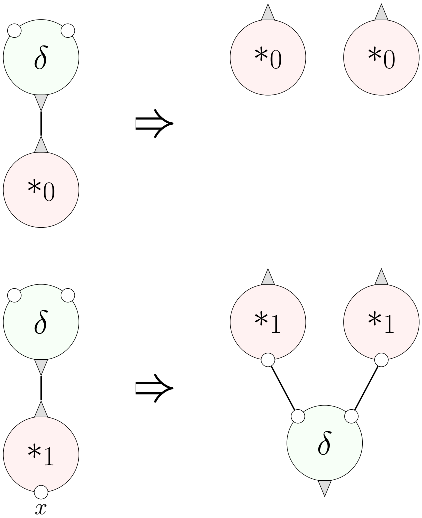
No normal form $\implies$ no result



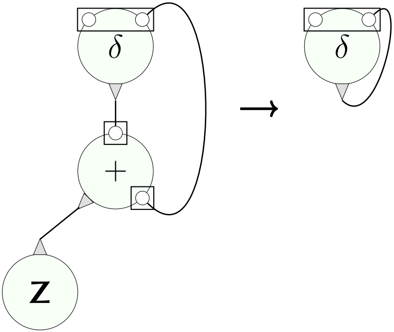
Vicious circles



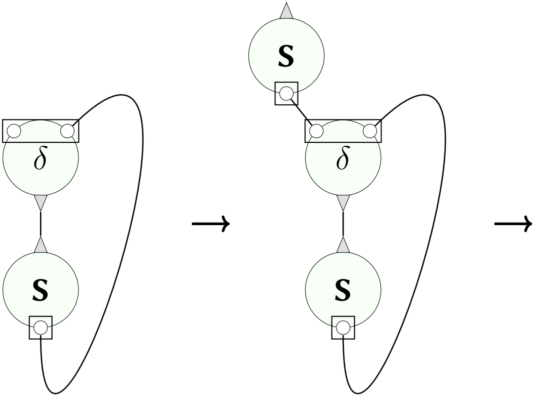
δ rules



Vicious circles: avoid with semi-simple nets

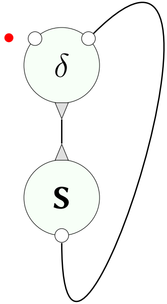


Semi-simple nets are overly restrictive

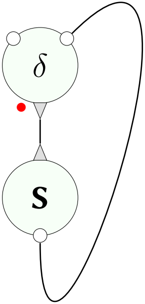


Equivalent to TRS $x \to Sx$

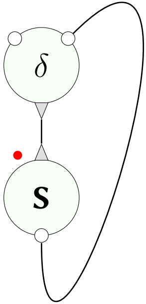
Previous approach: stream



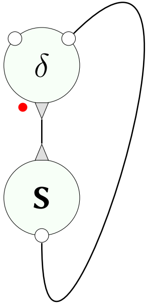
Previous approach: stream



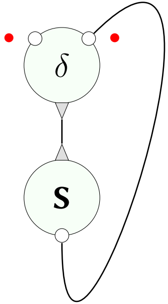
Previous approach: stream



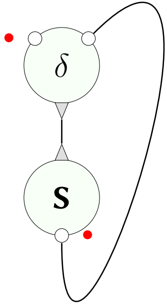
Previous approach: stream



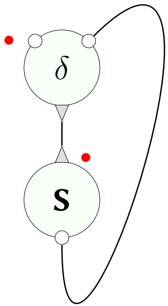
Previous approach: stream



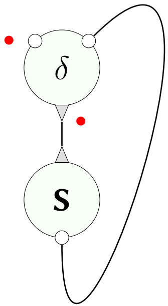
Previous approach: stream



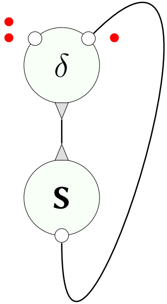
Previous approach: stream



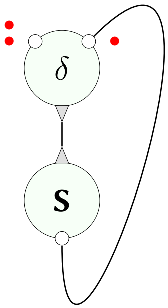
Previous approach: stream



Previous approach: stream – success!

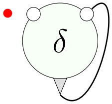


Previous approach: stream – (partial) success!



How know this is a stream?

Previous approach: vicious circle



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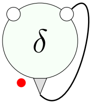
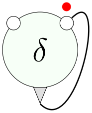
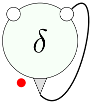


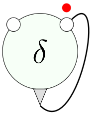
Previous approach: vicious circle



Previous approach: vicious circle



Previous approach: vicious circle – (partial) failure!

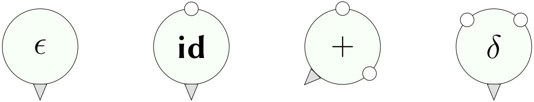


Track path...?

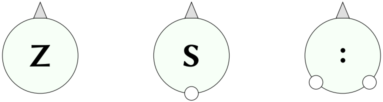
Algebraic approach

Function-constructor nets

Functions: principal port input ; 0 or more outputs:



Constructors: principal port output; 0 or more inputs



Implications

Node input/output distinction $\implies$ net input/output interfaces.

Functions are not first-class citizens!

Value of net at port x_i is sequence of constructors at that port.
$obs_{x_i}(N) \in \mathcal{N}_{\mathcal{C}}^{\infty}$

Value is observational at output ports, not normal form.
Value need not be finite.

Domain $\neq$ co-domain.

Evaluation: $\mathcal{N}_{\Sigma} \Downarrow_{\mathcal{R}} \mathcal{P} \times \mathcal{N}_{\mathcal{C}}^{\infty}$ (total) ; $\mathcal{P} =$ set of output port labels

$$v_{x_i}(N) = \texttt{lfp}(obs_{x_i}(\to^{\infty} (N)))$$

If $N \Downarrow V$ then $N \simeq V$.
If $N_1 \Downarrow V$ and $N_2 \Downarrow V$ then $N_1 \simeq N_2$.
$\simeq$ is an **equivalence relation**.

Textual syntax for nets

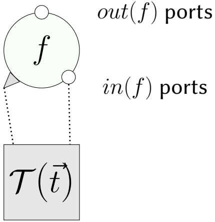
Grammar:

$$t, u ::= \bullet \mid x \mid f(\vec{t}) \mid C(\vec{t}) \mid t \sim u^{\vec{x}} \mid t \| u$$

Translation:

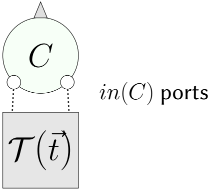
- $\bullet$ is the empty net.
- x is a wire labelled x at its input end.

- $f(\vec{t})$ is a function node with each input port is connected to one element of $\vec{t}$.

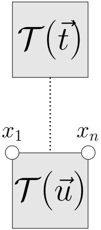


Textual syntax for nets II

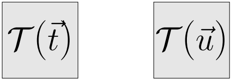
- $C(\vec{t})$ is a constructor node with input $\vec{t}$.



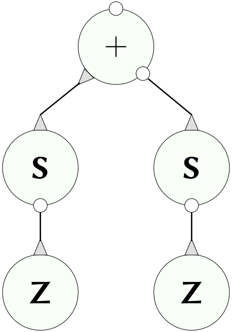
- $t \sim u^{\vec{x}}$ is u with output ports labelled $\vec{x}$ connected to a t or to u.



- $t \| u$ is the juxtaposition of the translated nets t and u.



Textual syntax for nets – example



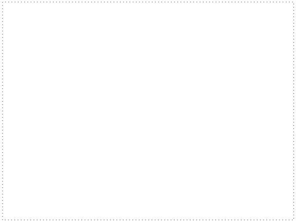
$$+(S(Z), S(Z))$$

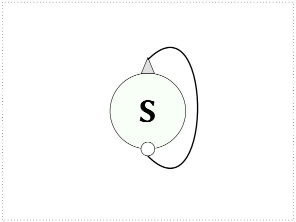
VS (INPLA)

```
+(r,y)~S(x), y~S(y1), x~Z, y1~Z;
```

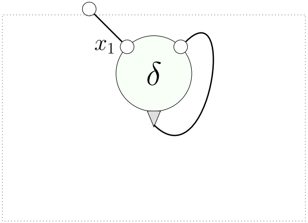
Net value equivalence classes

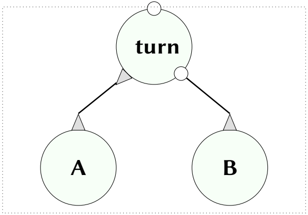
$V = \varnothing$





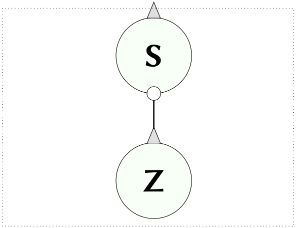
$V = \{(1, [\,])\}$ or just $[\,]$

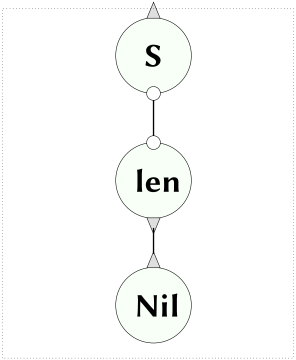




Net value equivalence classes

$V = S(Z)$

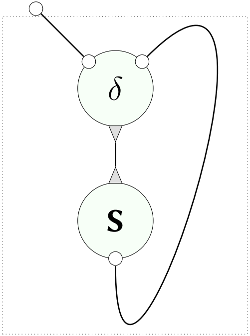


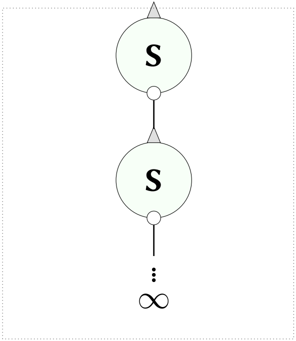


In general, $V = C_i(C_{i+1}(\ldots(\ldots))\ldots)$

Net value equivalence classes

$V = S^\infty$





In general, $V = C_i(\ldots(C_n)\ldots)^\infty$.

Productivity

$V = \varnothing$ or $V = [\ ] \implies 0/$**un -productive**
$|V| = 0$

$V = C_i(C_{i+1}(\ldots(\ldots))\ldots) \implies n/$**finitely -productive**
$|V| = \max(n)$

$V = C_i(\ldots(C_n)\ldots)^\infty \implies \infty/$**stream -productive**
$|V| = \infty$

$N \cong V \implies |N| = |V|$

Calculating productivity of nets

Concrete domain: $\mathcal{N}_\Sigma$ where input size $\implies$ output size (**data oblivious**).
Abstract domain: $\mathbb{N}^\infty_\bot = \mathbb{N}^+ \cup \{\bot, \infty\}$.

For term t we define **productivity*** **of** t, $|t|$ as:

- $| \bullet | = \bot$
- $| x | = \bot$
- For any nullary constructor, C, $| C | = 1$
  For a unary constructor, $| C(x) | = 1 \oplus | x |$
  For a binary constructor, $| C(x_1, x_2) | = 1 \oplus (| x_1 |, | x_2 |)$
- For a function f, $|f(t)| = |f|(|t|)$
- $|t_1 \sim t_2^{\vec{x}}| = |t_1[|t_2|/|\vec{x}|]|$, where $[\ldots]$ denotes substitution
- $|t_1 \| t_2| = (|t_1|, |t_2|)$.

$\bot$ denotes unproductive ($\forall n \in \mathbb{N}^\infty . \bot < n$).
∞ denotes stream productive ($\forall n \in \mathbb{N}^+_\bot . n < \infty$).

(*Should be productivity of t at port p, $|t|_p$.)

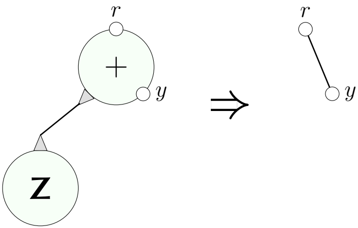
Algebra

$$\mathcal{A} = (\mathbb{N}^{\infty}_{\perp}, \oplus)$$

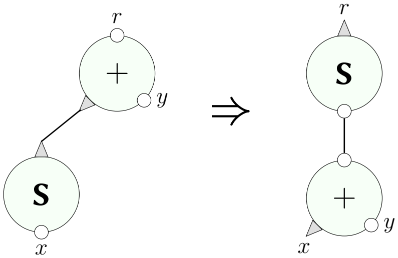
Addition: $\oplus$

- $n \oplus m = n + m \quad n, m \in \mathbb{N}^{+}$
- $n \oplus \perp = n \oplus \perp$
- $\perp \oplus n = \perp$
- $\infty \oplus n = n \oplus \infty = \infty$

Productivity rule example – addition



$$+(Z, y) \Rightarrow y$$



$$+(S(x), y) \Rightarrow S(+(x, y))$$

Productivity rule example – addition

$$+(Z, y) \Rightarrow y$$

$$|+|(|Z|, |y|) = |y|$$
$$|+|(1, |y|) = |y|$$

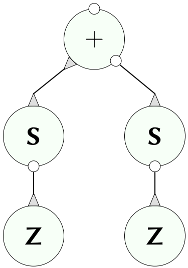
$$+(S(x), y) \Rightarrow S(+(x, y))$$

$$|+|(|S(x)|, |y|) = |S(+(x, y))|$$
$$|+|(1 \oplus |x|, |y|) = 1 \oplus (|+|(|x|, |y|))$$

Least fixed point is:

$$|+|(m, n) = m \dot{-} 1 \oplus n$$

Productivity rule example – addition

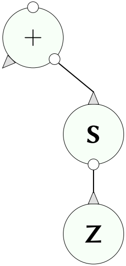


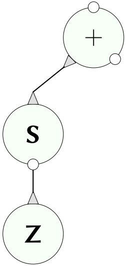
$$+(S(Z), S(Z))$$

$$\begin{aligned} |+(S(Z), S(Z))| &= |+|(|S(Z)|, |S(Z)|) \\ &= |+|(2, 2) \\ &= 2 \dot{-} 1 \oplus 2 \\ &= 3 \end{aligned}$$

Check: $+(S(Z), S(Z)) \simeq S(S(Z))$
$|S(S(Z))| = 3$

Productivity rule example – why ⊥ not 0





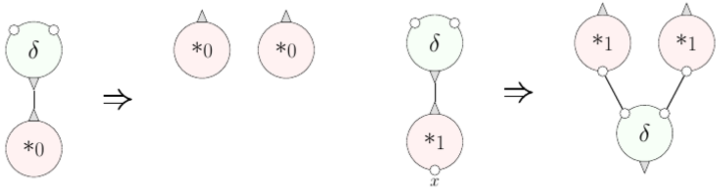
$$+(\bullet, S(Z))$$

$$\begin{aligned}|+(\bullet, S(Z))| &= \bot \dot{-} 1 \oplus 2 \\ &= \bot \\ &\cong 0\end{aligned}$$

$$+(S(Z), \bullet)$$

$$\begin{aligned}|+(S(Z), \bullet)| &= 2 \dot{-} 1 \oplus \bot \\ &= 1 \oplus \bot \\ &\cong 1\end{aligned}$$

Duplication

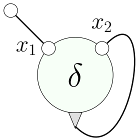


Interaction rule:	$\delta(Z) \Rightarrow Z \parallel Z$
Productivity equation:	$\lvert\delta\rvert(1) = (1, 1)$

Interaction rule:	$\delta(S(x)) \Rightarrow S(x_1) \parallel S(x_2) \sim \delta^{x_1,x_2}(x)$
Productivity equation:	$\lvert\delta\rvert(1 \oplus \lvert x\rvert) = (1 \oplus \pi_1(y), 1 \oplus \pi_2(y))$
	where $y = \lvert\delta^{x_1,x_2}\rvert(\lvert x\rvert)$

Closed form solution: $\lvert\delta\rvert(n) = (n, n)$.

Vicious circles

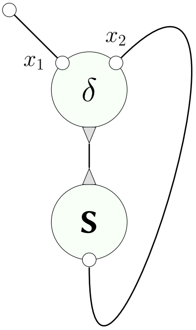


$$x_1 \sim \delta^{x_1,x_2}(x_2)$$

$$\begin{aligned} |x_1 \sim \delta^{x_1,x_2}(x_2)| &= \pi_1(|\delta^{x_1,x_2}(x_2)|) \\ &= |x_1 \sim \delta^{x_1,x_2}(x_2)| \end{aligned}$$

What's the lfp of $|t| = |t|$?
$|t| = \bot$

Streams



$$x_1 \sim \delta^{x_1,x_2}(S(x_2))$$

$$\begin{aligned} |x_1 \sim \delta^{x_1,x_2}(S(x_2))| &= \pi_1(|\delta^{x_1,x_2}(S(x_2))|) \\ &= \pi_1(|S(x_1) \sim \delta^{x_1,x_2}(S(x_2))|, |\ldots|) \\ &= |S(x_1) \sim \delta^{x_1,x_2}(S(x_2))| \\ &= 1 \oplus |x_1 \sim \delta^{x_1,x_2}(S(x_2))| \end{aligned}$$

What's the lfp of $|t| = 1 \oplus |t|$?
$|t| = \infty$

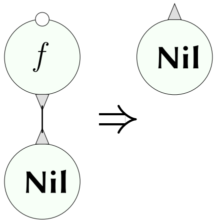
Do solutions always exist?

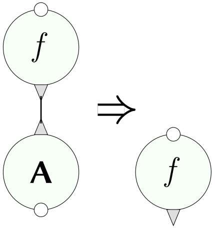
Yes because lfp of $|\cdot|$ always exists because:

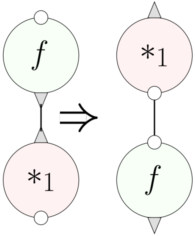
- $(\mathbb{N}_\perp^\infty, \sqsubseteq)$ is a dcpo with least element:
  - $\sqsubseteq = \leq$ extended for $\perp, \infty$.
  - Every directed subset of $\mathbb{N}_\perp^\infty$ has lub.
- $\oplus$ is Scott-continuous.

Hence by Kleene fixed-point theorem, lfp exists and equals $\sup\{\, |\cdot|^n(\perp) \mid n \in \mathbb{N}\,\}$.

Beyond data obliviousness

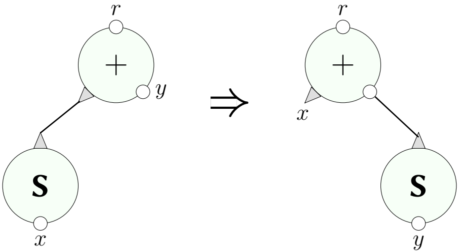






What to do with accumulating parameters? (weak convergence)

$$+(S(x), y) \Rightarrow +(x, S(y))$$



$$|+|(1 + |x|, |y|) = |+|(|x|, 1 + y)$$

$$\begin{aligned} |+|(|S^{\infty}|, |y|) &= |+|(|S^{\infty}|, 1 + |y|) \\ &= |+|(\infty, 1 + |y|) \end{aligned}$$

lfp of $|f|$ when $|f|(n, m) = |f|(n, m')$ is $\bot$.
So $|+|(\infty, |y|) = \bot$.